Assignment 12.

Laurent Series. Isolated singular points.

This assignment is due Wednesday, April 22. Collaboration is welcome. If you do collaborate, make sure to write/type your own paper.

NOTATION. We sometimes use $\sum_{\mathbb{Z}}$ instead of $\sum_{n=-\infty}^{\infty}$.

(1) Suppose f has a Laurent expansion

$$f(z) = \sum_{\mathbb{Z}} a_n (z - z_0)^n$$

in an annulus $r < |z - z_0| < \infty$. Prove that $f(z)$ can be also represented in the form

$$f(z) = \sum_{\mathbb{Z}} \tilde{a}_n z^n$$

in an annulus $\tilde{r} < |z| < \infty$.

(*Hint:* Use Laurent series theorem. Another hint is that *almost no* calculations are needed here.)

COMMENT. This problem justifies use of the term "Laurent expansion at ∞" without specifying center z_0, since by the statement above we can choose z_0 to be 0.

(2) Expand each of the following functions in a Laurent series at the indicated points:
(a) $\frac{1}{z^2+1}$ at $z = i$ and $z = \infty$,
(b) $z^2 e^{1/z}$ at $z = 0$ and $z = \infty$.

(3) Find and classify singular points (i.e. in each case decide whether the point is removable, a pole of order N, essential, or not an isolated singular point), including infinity, of the following functions:
(a) $\frac{1}{z-z^3}$, (b) $\frac{1}{(z^2+4)^2}$, (c) $\frac{e^z}{1+z^2}$, (d) $\frac{z^2+1}{e^z}$, (e) $\frac{1}{e^z-1} - \frac{1}{z}$,
(f) e^{-1/z^2}, (g) $\tan z$, (h) $\cot \frac{1}{z}$, (i) $\cot \frac{1}{z} - \frac{1}{z}$.

(4) Suppose z_0 is an isolated singular point of the function f of a given type (removable, pole of order N, essential). Show that z_0 is an isolated singular point of
(a) $g(z) = 1/f(z)$,
(b) $h(z) = f^2(z)$
and find its type.